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Cauchy's Problem for Almost Linear Elliptic Equations in Two Independent Variables

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I. INTRODUCTION

It is well-known (cf. [4] p. 108) that due to its unstable nature the Cauchy problem for elliptic partial differential equations is an improperly posed problem in the sense of Hadamard. Nevertheless, situations arise in mathematical physics for which it becomes necessary to solve such a problem, in particular when it is desired to construct an inverse solution to what is essentially a free boundary problem ([3]). In such cases the differential equation and prescribed data are often analytic and, hence, permit an application of the Cauchy-Kowalewski theorem. This approach is not very satisfactory, however, since what is actually required is a method that can be adapted for numerical integration. For the case of quasilinear equations in two independent variables (x, y), Garabedian has introduced a method which overcomes this difficulty by using characteristic coordinates to reduce the differential equation to a canonical system and then solving a one parameter family of related (stable) hyperbolic Cauchy problems ([3], [4] p. 623–633). In this paper we present a new method for solving the Cauchy problem for the case of almost-linear elliptic equations in a manner that is suitable for numerical computation. This method is based on the use of conjugate coordinates and reduces the Cauchy problem to finding a fixed point of a contraction mapping.

II. CONJUGATE COORDINATES AND THE CAUCHY PROBLEM

We seek a solution of the almost linear elliptic partial differential equation (written in normal form)

$$u_{xx} + u_{yy} = g(x, y, u, u_x, u_y) \tag{1}$$

which satisfies the Cauchy data

$$\begin{aligned} u(x, y) &= \Phi(x + iy), \qquad x + iy \in L, \\ \frac{\partial u(x, y)}{\partial n} &= \Omega(x + iy), \qquad x + iy \in L, \end{aligned} \tag{2}$$

where L is a given analytic arc, n is the unit outward normal to L and g, Φ, and Ω are assumed to have certain regularity properties to be described shortly. By the use of a conformal transformation, we can assume without loss of generality that the arc L is in fact a segment of the x axis containing the origin (i.e., $y = 0$ in Eq. (2)). By introducing *conjugate coordinates* ([5], [6])

$$\begin{aligned} z &= x + iy, \\ z^* &= x - iy, \end{aligned} \tag{3}$$

Eq. (1) becomes an equation of hyperbolic form:

$$U_{zz^*} = f(z, z^*, U, U_z, U_{z^*}), \tag{4}$$

where

$$u\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right) = U(z, z^*),$$

and the Cauchy data is transformed into

$$\begin{aligned} U(z, z^*) &= \Phi(z) \qquad \text{on } z = z^*, \\ \frac{\partial U(z, z^*)}{\partial z} - \frac{\partial U(z, z^*)}{\partial z^*} &= -i\Omega(z) \qquad \text{on } z = z^*. \end{aligned} \tag{5}$$

We assume at this point that as a function of its first two arguments, $f(z, z^*, \xi_1, \xi_2, \xi_3)$ is holomorphic in a bicylinder $G \times G^*$, where $G^* = \{z \mid z^* \in G\}$, and G is simply connected, and as a function of its last three variables it is holomorphic in a sufficiently large ball about the origin. We further assume that G contains the origin and is symmetric with respect to conjugation, i.e., $G = G^*$, and that $\Phi(z)$ and $\Omega(z)$ are holomorphic for all $z \in G$. The domain G described above is known as a *fundamental domain* ([5], [6]).

Now suppose $U(z, z^*)$ is a solution of Eq. (4) which is bounded and holomorphic in $G \times G^*$ and define a new function $s(z, z^*) = U_{zz^*}(z, z^*)$. It then follows that

$$U(z, z^*) = \int_0^z \int_0^{z^*} s(\xi, \xi^*)\, d\xi^*\, d\xi + \int_0^z \varphi(\xi)\, d\xi + \int_0^{z^*} \psi(\xi^*)\, d\xi^* + U(0, 0), \tag{6}$$

$$U_z(z, z^*) = \int_0^{z^*} s(z, \xi^*)\, d\xi^* + \varphi(z), \tag{7}$$

$$U_{z^*}(z, z^*) = \int_0^z s(\xi, z^*)\, d\xi + \psi(z^*), \tag{8}$$

where $\varphi(z) = U_z(z, 0)$ and $\psi(z^*) = U_{z^*}(0, z^*)$. Note that $s(z, z^*)$ must satisfy the equation

$$\begin{aligned} s(z, z^*) = f\Big[z, z^*, &\int_0^z \int_0^{z^*} s(\xi, \xi^*)\, d\xi^*\, d\xi + \int_0^z \varphi(\xi)\, d\xi + \int_0^{z^*} \psi(\xi^*)\, d\xi^* \\ &+ U(0, 0), \int_0^{z^*} s(z, \xi^*)\, d\xi^* + \varphi(z), \int_0^z s(\xi, z^*)\, d\xi + \psi(z^*)\Big] \end{aligned} \tag{9}$$

and, conversely, if $s(z, z^*)$ satisfies (9) then a solution of (4) is given by (6). The initial conditions (5) become

$$\int_0^z \int_0^z s(\xi, \xi^*)\, d\xi^*\, d\xi + \int_0^z \varphi(\xi)\, d\xi + \int_0^z \psi(\xi^*)\, d\xi^* + U(0, 0) = \Phi(z) \tag{10}$$

or, differentiating in the z plane,

$$\int_0^z s(z, \xi^*)\, d\xi^* + \int_0^z s(\xi, z)\, d\xi + \varphi(z) + \psi(z) = \Phi'(z) \tag{11}$$

and

$$\int_0^z s(z, \xi^*)\, d\xi^* + \varphi(z) - \int_0^z s(\xi, z)\, d\xi - \psi(z) = -i\Omega(z). \tag{12}$$

Equations (11) and (12) now yield the following expressions for $\varphi(z)$ and $\psi(z)$ in terms of the function $s(z, z^*)$:

$$\varphi(z) = \tfrac{1}{2}[\Phi'(z) - i\Omega(z)] - \int_0^z s(z, \xi^*)\, d\xi^*, \tag{13}$$

$$\psi(z) = \tfrac{1}{2}[\Phi'(z) + i\Omega(z)] - \int_0^z s(\xi, z)\, d\xi. \tag{14}$$

Hence, we can express the functions $\varphi(z)$ and $\psi(z)$ as operators on the function $s(z, z^*)$. In particular, if we define the operators B_i, $i = 1, 2, 3$, by the right sides of (6), (7), and (8), respectively, where $\varphi(z)$ and $\psi(z)$ are determined from Eq. (13) and (14) (note that $U(0, 0) = \Phi(0)$), then $s(z, z^*)$ satisfies the equation

$$s(z, z^*) = f(z, z^*, B_1[s(z, z^*)], B_2[s(z, z^*)], B_3[s(z, z^*)]). \tag{15}$$

III. The Solution of Cauchy's Problem

The approach to be used in this section is patterned after the ideas of [1], and [2] (see also [5] p. 154–164). Consider the class $HB(\Delta\rho, \Delta\rho^*)$ of functions of two complex variables which are holomorphic and bounded in $\Delta\rho \times \Delta\rho^*$, where $\Delta\rho = \{z \mid |z| < \rho\}$, $\Delta\rho^* = \{z \mid z^* \in \Delta\rho\}$. If a norm is defined on $HB(\Delta\rho, \Delta\rho^*)$ by

$$\| s \|_\lambda = \sup\{e^{-\lambda(|z|+|z^*|)} \mid s(z, z^*)|\}, \tag{16}$$

where $(z, z^*) \in \Delta\rho \times \Delta\rho^*$ and $\lambda > 0$ is fixed, $HB(\Delta\rho, \Delta\rho^*)$ becomes a Banach space which we denote $A\rho$. We shall now show that the operator T defined by

$$Ts(z, z^*) = f(z, z^*, B_1[s(z, z^*)], B_2[s(z, z^*)], B_3[s(z, z^*)]) \tag{17}$$

maps a closed ball of the Banach space $A\rho$ into itself, and is a contraction mapping, thus providing a constructive method for obtaining the unique solution to our Cauchy problem.

By hypothesis, f is holomorphic in a compact subset of the space of five complex variables and, hence, from Schwarz's lemma for functions of several complex variables ([5] p. 38, 159), a Lipschitz condition holds there with respect to the last three arguments, i.e.,

$$\begin{aligned} &| f(z, z^*, \xi_1, \xi_2, \xi_3) - f(z, z^*, \xi_1^0, \xi_2^0, \xi_3^0)| \\ &\qquad \leqslant C_0\{| \xi_1 - \xi_1^0 | + | \xi_2 - \xi_2^0 | + | \xi_3 - \xi_3^0 |\}, \end{aligned} \tag{18}$$

where C_0 is a positive constant. Hence, for $s_1, s_2 \in A\rho$ and ρ sufficiently small,

$$\| Ts_1 - Ts_2 \|_\lambda \leqslant C_0\{\| B_1 s_1 - B_1 s_2 \|_\lambda + \| B_2 s_1 - B_2 s_2 \|_\lambda + \| B_3 s_1 - B_3 s_3 \|_\lambda\}. \tag{19}$$

From estimates of the form

$$\left| \int_0^z s(\xi, z^*)\, d\xi \right| \leqslant \int_0^{|z|} \| s \|_\lambda\, e^{\lambda|\xi|+\lambda|z^*|} \, | d\xi | \leqslant \frac{1}{\lambda} e^{\lambda|z|+\lambda|z^*|} \, \| s \|_\lambda , \tag{20}$$

i.e.,

$$\left\| \int_0^z s(\xi, z^*)\, d\xi \right\|_\lambda \leqslant \frac{\| s \|_\lambda}{\lambda} \tag{21}$$

(where we have assumed $s(z, z^*)$ is regular in the polydisc $\Delta\rho \times \Delta\rho^*$, so that the curvilinear path of integration may be replaced by a straight line-segment) it can be seen that

$$\| B_i s_1 - B_i s_2 \|_\lambda \leqslant \frac{N_i}{\lambda} \| s_1 - s_2 \|_\lambda , \qquad i = 1, 2, 3, \tag{22}$$

where the N_i are positive constants independent of λ and $\lambda > 0$. Hence,

$$\| Ts_1 - Ts_2 \|_\lambda \leqslant \frac{M}{\lambda} \| s_1 - s_2 \|_\lambda , \tag{23}$$

where M is a positive constant independent of λ. Inequality (23) implies that

$$\| Ts \|_\lambda \leqslant \frac{M}{\lambda} \| s \|_\lambda + \| T_0 \|_\lambda < \frac{M}{\lambda} \| s \|_\lambda + M_0 , \tag{24}$$

where M_0 is a positive constant. Therefore, for $\| s \|_\lambda < M_0$ and λ sufficiently large, $\| Ts \|_\lambda < M_0$, i.e., T takes a closed ball in $A\rho$ into itself. Equation (23) also implies that, for λ sufficiently large,

$$\| Ts_1 - Ts_2 \|_\lambda < \| s_1 - s_2 \|_\lambda , \tag{25}$$

i.e., T is a contraction mapping. The existence and uniqueness of a solution to the equation $Ts = s$ in $A\rho$ is now immediate. We have proved the following:

THEOREM 1. *Let G be a fundamental domain for the elliptic equation* (1) *and let $f(z, z^*, \xi_1, \xi_2, \xi_3)$ be holomorphic in $G \times G^* \times B^{(3)}$, where $B^{(3)}$ is a sufficiently large ball about the origin. Assume, further, that $G = G^*$ and the functions $\Phi(z)$, $\Omega(z)$ are holomorphic in G. Then, for ρ sufficiently small, Eq.* (17), (13), (14), *and* (6) *provide a constructive method for obtaining a unique solution of Eq.* (1) *in $| z | \leqslant \rho$, satisfying the Cauchy data* (2).

It is important to note here that the unstable dependence of the solution of the elliptic equation (1) on the (real) Cauchy data (2) appears exclusively in the step where this data is extended to complex values of the independent variable x. When this can be done in an elementary way, for example, by direct substitution via the transformation (3), no instabilities will occur when one uses the contraction mapping operator T to obtain approximations to the desired solution.

For the case where Eq. (1) is linear, Henrici ([5], [6]) has used conjugate coordinates and the Riemann function to obtain a solution of Cauchy's problem. Hence, Theorem 1 can be considered as an extension of Henrici's results to the case of almost linear elliptic equations.

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